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LETTER TO THE EDITOR

Equivalence and solution of anisotropic spin-1 models and generalized t–J fermion models in one dimension*

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Abstract. We study the relationship of two 'q-deformed' spin-1 chains—both of them are solvable models—with a generalized supersymmetric t–J fermion model in one dimension. One of the spin-1 chains is an anisotropic VBS model for which we calculate ground state and ground-state properties. The other spin-1 chain corresponds to the Zamolodchikov–Fateev model which is solvable by Bethe ansatz and is equivalent to a certain t–J model. The two spin-1 models intersect for a certain value of the 'deformation' parameter q in a second-order phase transition.

Recently the t–J model has attracted some interest in connection with high-temperature superconductivity (Anderson 1987, Zhang and Rice 1988). This model describes hard-core fermions with nearest-neighbour hopping and spin exchange interaction along a linear chain:

$$H_{tJ} = \sum_{j=1}^{L} \left\{ -t\mathscr{P} \sum_{\sigma=\pm} (c_{j+1\sigma}^{+} c_{j\sigma} + \mathrm{h.c.})\mathscr{P} + J(\boldsymbol{s}_j \cdot \boldsymbol{s}_{j+1} - \tfrac{1}{4} n_j n_{j+1}) \right\} \tag{1}$$

where $\mathscr{P}$ is the projector on the subspace of non-doubly occupied states. With $\sigma = \pm$ we have the occupation numbers $n_{j\sigma} = c_{j\sigma}^{+} c_{j\sigma}$, $n_j = n_{j+} + n_{j-}$, and the spin-$\frac{1}{2}$ operators $\boldsymbol{s}_j$. Here the henceforth we assume periodic boundary conditions and that the number of sites L is even (for convenience).

At $J = 2t$ the Hamiltonian (1) becomes supersymmetric (Wiegmann 1988) and exactly solvable by Bethe ansatz (Schlottmann 1987, Bares and Blatter 1990). It also can be mapped on the Lai–Sutherland model (Lai 1974, Sutherland 1975) as shown by Sarkar (1990). In the following we shall show that the supersymmetric t–J model can also be related to a certain spin-1 Hamiltonian thus providing an integrable generalization.

We consider the so-called q-deformed spin-1 model which is based on the $\mathrm{U}_q[\mathrm{SU}(2)]$ quantum algebra (Jimbo 1985). Following Batchelor *et al* (1990) we write

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its Hamiltonian as

$$H(a,b;q)=b\sum_{j=1}^{L}\{a(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1})+[\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}+\tfrac{1}{2}(1-a)(q+q^{-1}-2)(S_j^zS_{j+1}^z)$$
$$+\tfrac{1}{4}(1+a)(q-q^{-1})(S_{j+1}^z-S_j^z)]^2+\tfrac{1}{4}a(1-a)(q+q^{-1}-2)^2(S_j^zS_{j+1}^z)^2$$
$$+\tfrac{1}{4}a(1+a)(q-q^{-1})(q+q^{-1}-2)(S_j^zS_{j+1}^z)(S_{j+1}^z-S_j^z)$$
$$+\tfrac{1}{4}(q-q^{-1})^2[(a-1+\tfrac{1}{2}(1+a)^2)S_j^zS_{j+1}^z+2(a+\tfrac{1}{8}(1+a)^2)((S_j^z)^2+(S_{j+1}^z)^2)]$$
$$+(a-1)+\tfrac{1}{2}a(q^2-q^{-2})(S_{j+1}^z-S_j^z)\}. \tag{2}$$

The $\boldsymbol{S}_j$ are the spin-1 operators and a and b are interaction constants. At $q=1$ the Hamiltonian reduces to the well known bilinear-biquadratic spin-1 model with o(3)-symmetry in spin-space, while for general complex q the model is 'deformed' in z-direction. In (2) the last term drops out in the whole sum, but has been added to the nearest-neighbour local interaction for convenience.

For several special values of a and b the model becomes exactly solvable or certain ground-state properties can be calculated exactly. We shall consider two such cases.

(α) The first case is the Hamiltonian limit of a 2D classical 19-vertex model which was solved by Zamolodchikov and Fateev (1980). This ZF model is obtained from (2) by setting $b=+a=1$. Thus we have

$$H_{\mathrm{ZF}}\equiv H(-1,1;q)$$
$$=\sum_{j=1}^{L}\{-\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}+[\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}+(q+q^{-1}-2)S_j^zS_{j+1}^z]^2$$
$$-\tfrac{1}{2}(q+q^{-1}-2)^2(S_j^zS_{j+1}^z)^2-\tfrac{1}{2}(q-q^{-1})^2[S_j^zS_{j+1}^z+(S_j^z)^2+(S_{j+1}^z)^2]-2\} \tag{3}$$

and the last term in (2) has been left out. The model is physically realistic, i.e. its Hamiltonian is Hermitian, for real q and unimodular q ($|q|=1$). In the latter case the model is critical, i.e. there is no gap in the excitation spectrum (Babujian and Tsvelick 1985, Kirillov and Reshetikhin 1987, Alcaraz and Martins 1989).

(β) The second case is the (generalized) q-deformed VBS model (Batchelor *et al* 1990) which is obtained from (2) by setting

$$a=1+q^2+q^{-2}\qquad b=2[(q^2+q^{-2})(1+q^2+q^{-2})]^{-1}$$

That is,

$$H_{\mathrm{VBS}}^{q}\equiv H(1+q^2+q^{-2},2[(q^2+q^{-2})(1+q^2+q^{-2})]^{-1};q). \tag{4}$$

Below we shall show that ground state and certain ground-state properties can be determined for general q in the same way as for the special isotropic VBS model at $q=1$, considered by Affleck *et al* (1988). Note from (3) and (4) that ZF and VBS coincide at $q=\pm\mathrm{i}$.

To treat the VBS model (a more detailed version will be published elsewhere) one introduces the eigenstates $|\pm\rangle$, $|0\rangle$ for $S_j^z=\pm1$, 0 on each site and defines the nine possible combined neighbouring states at sites j, $j+1$ as: $|1\rangle=|++\rangle$, $|2\rangle=|+0\rangle$, $|3\rangle=|0+\rangle$, $|4\rangle=|+-\rangle$, $|5\rangle=|00\rangle$, $|6\rangle=|-+\rangle$, $|7\rangle=|0-\rangle$, $|8\rangle=|-0\rangle$, $|9\rangle=|--\rangle$. Then one shows that (4) can be written as

$$H_{\mathrm{VBS}}^{q}=2\sum_{j=1}^{L}h_{j,j+1} \tag{5}$$

such that the local interaction h is the sum of projectors, $h=P_1+P_9+\bar{P}_{23}+\bar{P}_{78}+\bar{P}_{456}$, which project precisely on the five states of the 'q-quintet':

$$|1\rangle \qquad |9\rangle \qquad |2\rangle+q^2|3\rangle \qquad |7\rangle+q^2|8\rangle \qquad q^{-2}|4\rangle+q^2|6\rangle+(q+q^{-1})|5\rangle \tag{6a}$$

while annihilating the four states of the 'q-triplet' and 'q-singlet':

$$q^{22}\rangle-|3\rangle \qquad q^2|7\rangle-|8\rangle \qquad |4\rangle-|6\rangle+(q-q^{-1})|5\rangle \qquad q|4\rangle+q^{-1}|6\rangle-|5\rangle \tag{6b}$$

or any linear combination of them. The notation here indicates that these states reduce to the usual quintet, triplet and singlet states in the case of O(3) symmetry at $q=1$.

For general q the projectors P and thus h are not Hermitian. However, it is easy to define a physically realistic VBS model by setting

$$\tilde{H}^q_{\mathrm{VBS}}=2\sum_{j=1}^{L} h^+_{j,j+1}h_{j,j+1}. \tag{7}$$

Of course, h^+h is now Hermitian acting on the states (6b) in the same way as h. For real q and $q=\pm\mathrm{i}$ we have $\tilde{H}^q_{\mathrm{VBS}}=H^q_{\mathrm{VBS}}$.

The VBS ground state is given by the ansatz

$$|\psi_0\rangle:=\operatorname{Tr} g_1\otimes g_2\otimes\ldots\otimes g_L \tag{8}$$

where at site j define the matrix

$$g:=\begin{pmatrix} q^{-1}|0\rangle & -\sqrt{q+q^{-1}}|+\rangle \\ \sqrt{q+q^{-1}}|-\rangle & -q|0\rangle \end{pmatrix} \tag{9}$$

in terms of S^z_j eigenstates. To see that $\tilde{H}^q_{\mathrm{VBS}}|\psi_0\rangle=0$ one considers the product of nearest neighbour sites

$$g_j\otimes g_{j+1}\Rightarrow g\otimes g=\begin{pmatrix} q^{-2}|5\rangle-(q+q^{-1})|4\rangle & \sqrt{q+q^{-1}}[q|2\rangle-q^{-1}|3\rangle] \\ -\sqrt{q+q^{-1}}[q|7\rangle-q^{-1}|8\rangle] & q^2|5\rangle-(q+q^{-1})|6\rangle \end{pmatrix}. \tag{10}$$

The entries are linear combinations of the states (6b) and thus $(h^+h)g\otimes g=0$, i.e. $\tilde{H}^q_{\mathrm{VBS}}|\psi_0\rangle=0$. $|\psi_0\rangle$ is ground state as $\tilde{H}^q_{\mathrm{VBS}}\geqslant 0$. One can show that $|\psi_0\rangle$ is unique except for the critical case $q=\pm\mathrm{i}$ where $|\psi_0\rangle=|0\,0\,0\ldots0\rangle$ becomes degenerate with other states.

Correlation functions within the ground state can be calculated in a straightforward way. While the derivation and the results will be published elsewhere we here only mention a few typical results. The longitudinal correlation $\langle S^z_1S^z_r\rangle$ between sites $j=1$ and $j=r$ is given for real and unimodular q, respectively, by

$$\langle S^z_1S^z_r\rangle=-\begin{cases} (q^2+q^{-2}+2)[-(1+q^2+q^{-2})]^{-r} & q \text{ real} \\ \dfrac{4\cos^2\varphi}{[1-2|\cos\varphi|]^2}\left(\dfrac{1-2|\cos\varphi|}{1+2|\cos\varphi|}\right)^r & q=\mathrm{e}^{\mathrm{i}\varphi} \end{cases} \tag{11}$$

while the transversal correlation $\langle S^x_1S^x_r\rangle$ is obtained as

$$\langle S^x_1S^x_r\rangle=-\begin{cases} (q^3+q^{-3}+2)[-(1+q^2+q^{-2})]^{-r} & q \text{ real} \\ 2[-(1+2|\cos\varphi|)]^{-r}\cos(2(r-1)\varphi)[\cos\varphi+|\cos\varphi|] & q=\mathrm{e}^{\mathrm{i}\varphi}. \end{cases} \tag{12}$$

Both show exponential decay with correlation lengths which only coincide for real q, but in general are different from one another (here for $q=\mathrm{e}^{\mathrm{i}\varphi}$). This indicates that the excitation spectrum of the Hamiltonian for general q is quite interesting and rich in structure. As noticed before the VBS model intersects the ZF model at $q=\pm\mathrm{i}$. There is a second-order phase transition at this point, the corresponding singularity is the square-root singularity of $\sqrt{q+q^{-1}}$ in the ground state (9), which is also seen in the occurrence of $|\cos\varphi|$ in equations (11) and (12).

Next we show that the ZF model (3) is equivalent to a 'q-deformed' t–J model with $t=1$, $J=2$ which is a generalization of the supersymmetric version of the fermionic Hamiltonian (1). We have

$$
\begin{aligned}
H(-1,1;q) \Leftrightarrow H^q_{tJ} & \\
:= \sum_{j=1}^{L} \Bigg\{ & -\mathscr{P} \sum_{\sigma=\pm} (c^+_{j+1,\sigma} c_{j\sigma} + \text{h.c.})\mathscr{P} + 2\boldsymbol{s}_j \cdot \boldsymbol{s}_{j+1} - \tfrac{1}{2} n_j n_{j+1} \\
& + (q+q^{-1})^2 [n_{j+} n_{j+1,-} + n_{j-} n_{j+1,+}] + \tfrac{1}{2}(q^2+q^{-2})(n_j + n_{j+1}) \\
& + (q+q^{-1})\mathscr{P} \sum_{\sigma=\pm} (c^+_{j\sigma} c^+_{j+1,-\sigma} + \text{h.c.})\mathscr{P} \Bigg\}.
\end{aligned}
\tag{13}
$$

The fermionic notation has been explained following equation (1). We remark that in (13) there appears a 'chemical potential' term in the third line and pair creation and annihilation terms on neighbouring sites in the last line.

To show the equivalence in (13) one identifies the local spin-1 states $|0\rangle$, $|\pm\rangle$ with the local fermion states: $|0\rangle$ for zero occupation, and $|\pm\rangle$ for one-fermion occupation with spin-$\pm\frac{1}{2}$, while the doubly occupied state is excluded in the model. It is then easy to show that the local interaction $h_{j,j+1}$ in (13) only has the following non-vanishing matrix elements:

$$
\begin{aligned}
&\langle \sigma\sigma | h | \sigma\sigma \rangle = -(q^2+q^{-2}) \\
&\langle \sigma, -\sigma | h | 00 \rangle = \langle 00 | h | \sigma, -\sigma \rangle = -(q+q^{-1}) \\
&\langle \sigma, -\sigma | h | \sigma, -\sigma \rangle = \langle \sigma, -\sigma | h | -\sigma, \sigma \rangle = 1 \\
&\langle 0\sigma | h | \sigma 0 \rangle = \langle \sigma 0 | h | 0\sigma \rangle = -1 \\
&\langle 0\sigma | h | 0\sigma \rangle = \langle \sigma 0 | h | \sigma 0 \rangle = -\tfrac{1}{2}(q^2+q^{-2})
\end{aligned}
\tag{14}
$$

where $|00\rangle$, $|0\sigma\rangle$, $|\sigma\sigma'\rangle$ etc denote the combined states at sites j and $j+1$. For the ZF model one shows that the local interaction in (3) has the same non-vanishing matrix elements, which then proves the equivalence in (13).

Finally we notice that at $q=\pm\mathrm{i}$ we recover from (13) the supersymmetric t–J model (1) with a chemical potential $\mu=2$, i.e. we have

$$H^{q=\mathrm{i}}_{tJ} \equiv H(-1,1;\mathrm{i}) = H_{tJ} + 2N \tag{15}$$

where N is the total particle number. We remark that the chemical potential renders the empty chain $|0\,0\,0\ldots 0\rangle$ an absolute ground state, while the interesting physics for the t–J model (without chemical potential) takes place in sectors with large particle numbers.

More detailed investigations of the models will be published elsewhere.

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